Week 10: First Derivative Test

Materials students should bring:
- Section 5.1 Lecture Notes

Before class:
- Bring a sign-in sheet to take role.
- Exam 2 packets should be in your mailbox before your meeting time. Please return them to the students at the end of this session.

Reminder: In the Lesson Plans, the **Chalkboard font** indicates things I hope you will write on the slide.

1. Remind students Excel 2 is due after Spring Break in Week 12 at the beginning of class.
2. Display Slide 1. Ask students to take out their 5.1 Lecture Notes page 6. Tell the students that this section is about connecting properties of the derivative to features of the graph. As a lead in, ask for a volunteer to estimate the derivative at $x = 2$ from the graph. Record the slope on the slide. Do the same at $x = 11$. Make notes on the slide that the derivative is positive when the function is increasing and the derivative is negative when the function is decreasing. Encourage your students to take notes.

3. Display Slide 2, and repeat that students should answer in coordinates. Remind them that they can refer to the similar problem on page 1 of the 5.1 Lecture Notes, which the large lecture professors discussed in class. Ask them to work with their groups to complete the table, and get their answers checked.
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4. Circulate and check their answers. Make sure to check the Key Observations at the bottom of the page, since it’s a little vague.

<table>
<thead>
<tr>
<th>Endpoints</th>
<th>(0,0) and (16,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critical points</td>
<td>(6,3), (10,5) and (12,1)</td>
</tr>
<tr>
<td>Singular points</td>
<td>none</td>
</tr>
<tr>
<td>Stationary points</td>
<td>(6,3), (10,5) and (12,1)</td>
</tr>
<tr>
<td>Relative maxima</td>
<td>(10,5) and (16,5)</td>
</tr>
<tr>
<td>Relative minima</td>
<td>(0,0) and (12,1)</td>
</tr>
<tr>
<td>Absolute maxima</td>
<td>(10,5) and (16,5)</td>
</tr>
<tr>
<td>Absolute minima</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

**Key observations**
- All maxima and minima occur at either a critical point or an end point.
- (Circle one) Critical points are always / not always relative maxima or minima.

**Note**
In this class, end points are relative extrema! (This agrees with our textbook.)

Give an example of an input $x$ at which the following hold:

| $g(x) > 0$ | Any $x \in (0,6) \cup (6,10) \cup (12,16)$ |
| $g(x) < 0$ | Any $x \in (10,12)$ |
| $g(x) = 0$ | $x = 6, 10, 12$ |

5. As they work (once everyone has copied down the questions on Slide 2), display Slide 3. Once you have checked a student’s answers, ask them to turn to Lecture Notes pages 7 and 8, and copy the missing functions into their notes. Ask them to begin work on the worksheet.

6. The worksheet has enough structure that students should be able to work through it with their group without much confusion. We hope it will provide enough scaffolding that students can complete the two first derivative test problems in the notes after the break.

7. The answers for the worksheet are on the next page. Please circulate and check their answers as they work. Once they are finished, invite them to take a break, returning to class for the second part of the Activity at a specified time. You need not collect the worksheet.
### Worksheet Answers

Given: \( h(x) = 3x^4 - 16x^3 - 18x^2 + 216x \) on the interval \([-5, \infty)\)

\[
h'(x) = 12x^3 - 48x^2 - 36x + 216 = 12(x - 3)^2(x + 2)
\]

1. Stationary points: Solve \( h'(x) = 0 \)
\[
h'(x) = (x - 3)^2(x + 2) = 0 \text{ so } x = -2, 3
\]

Singular points: Find \( x \)'s at which \( h'(x) \) does not exist.

**None.** This function is a polynomial (no denominator to make us divide by 0).

So the critical points for \( h(x) \) are \( x = -2 \) and \( x = 3 \).

2. Intervals: \([-5, -2), (-2, 3)\) and \((3, +\infty)\).

Check \( h'(x) \) is circled in #2.

She wants to use the First Derivative Test to find out whether \( h(x) \) is increasing or decreasing in each interval. So, she chooses a test point in each interval, and plugs it into \( \frac{h(x)}{h'(x)} \).

Test points can be any number inside the corresponding interval. The table shows one possibility.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sample Test Point</th>
<th>( h'(x) = -432 ) (negative)</th>
<th>( h(x) ) is Decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>([-5, -2)]</td>
<td>( x = -3 )</td>
<td>decreasing</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sample Test Point</th>
<th>( h'(0) = 216 )</th>
<th>( h(x) ) is Increasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-2, 3))</td>
<td>( x = 0 )</td>
<td>increasing</td>
<td></td>
</tr>
</tbody>
</table>

\[
\text{Check } h(x) \text{ is circled in #3.}
\]

3. From the table above, Shadra can see that she has relative minima at \( x = -2 \) and a relative maximum at \( x = -5 \). To find the \( y \)-coordinates, plug these \( x \)'s into (circle one) \( \frac{h(x)}{h'(x)} \). Find the \( y \)-values and write them in the table to the right.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>2345</td>
</tr>
<tr>
<td>-2</td>
<td>-328</td>
</tr>
</tbody>
</table>

**Conclusions:**

Interval(s) of increase: \((-2, \infty)\) Interval(s) of decrease: \([-5, -2)\).

Coordinates of: Relative minima: \((-2, -328)\) Relative maxima: \((-5, 2345)\)
8. Display Slide 4. Since their explanations are graded, we’d like the students to get their work checked carefully at a few (forgive me) critical points in the problem. Ask them to work through the two examples and ask to be checked as directed.


10. When a student is finished, return their Exam 2 and let them go. If students are still working, with 15 minutes left in the class, return the unclaimed exams, and display Slide 5. You don't need Slide 5 if they finish the work during class.

About the Exams
If they have questions that you can answer about their tests please do answer them, but if a student has a question about whether or not points were correctly deducted, please guide them to ask their professor. Encourage them to approach the large lecture professor if there is an issue, but avoid leading them to believe they will definitely get points back on a problem, in case there is something in the rubric that you don’t realize is there.

What we are grading for in First Derivative Test Problems
When we grade first derivative test problems, we are checking that the student
- Finds the critical numbers correctly,
- Uses the critical numbers to partition the domain into intervals,
- Chooses a test point in each interval,
- Evaluates the first derivative of the function at each test point*,
- Draws appropriate conclusions from this analysis about where the function increases and decreases and where the relative extrema lie, and
- Evaluates the function to find the (x,y)-coordinates of each relative extrema.

* Really, the student only needs to know the sign of the derivative at each test point, to determine if the function increases or decreases in the interval. To receive full credit, what they write must clearly reflect that they use the *derivative* (and not the original function) and an *appropriate test point* to determine the sign.

For example, say the student is studying a function \( f(x) \) on the interval \((-1, 4)\), and \( f'(2) = 3 \).

<table>
<thead>
<tr>
<th>( f ) is increasing on ((-1,4)) since it is positive.</th>
<th>( f ) is increasing on ((-1,4)) since ( f' ) is positive or ( f' = + )</th>
<th>( f ) is increasing on ((-1,4)) since ( f(2) = 59 ) or ( f(2) = + )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Does not receive full credit</td>
<td>Does not receive full credit</td>
<td>Does not receive full credit</td>
</tr>
<tr>
<td>( f ) is increasing on ((-1,4)) since ( f'(2) = 3 ).</td>
<td>( f ) is increasing on ((-1,4)) since ( f'(2) &gt; 0 )</td>
<td>( f ) is increasing on ((-1,4)) since ( f'(2) = + )</td>
</tr>
<tr>
<td>Receives full credit</td>
<td>Receives full credit</td>
<td>Receives full credit</td>
</tr>
</tbody>
</table>

Both the *test point* and the *function used to determine the sign* must be evident in the solution to receive full credit. The option with the bold border is the one the worksheet uses, and is the one that tends to leave the least ambiguity.
Solutions

Example 6
Use the first derivative test to find the relative maxima and minima of the function \( f(x) = 3x^4 + 5x^3 \) on the domain \((-\infty, \infty)\). Determine the intervals of increase and decrease on this domain. Complete the answer box, if there are no answers, write “none.”

\[ f(x) = 3x^4 + 5x^3 \text{ so } f'(x) = 12x^3 + 15x^2 = 3x^2(4x + 5) \]

Singular points: none. \( f'(x) \) is defined for all real numbers.

Stationary points: \( f'(x) = 3x^2(4x + 5) = 0 \) when \( x = 0 \) or \(-1.25\).

Finding coordinates:
\( x = -1.25: f(-1.25) \approx -2.44; \) coordinates \((-1.25, -2.44)\)
\( x = 0: f(0) = 0; \) coordinates \((0, 0)\)

Example 7
Use the first derivative test to find the relative maxima and minima of the function \( f(x) = 3x^4 + 4x^3 - 36x^2 + 10 \) on the domain \([1, \infty)\). Determine the intervals of increase and decrease on this domain. Complete the answer box, if there are no answers, write “none.”

\[ f(x) = 3x^4 + 4x^3 - 36x^2 + 10 \text{ so } f'(x) = 12x^3 + 12x^2 - 72x = 12x(x + 3)(x - 2) \]

Singular points: none. \( f'(x) \) is defined for all real numbers.

Stationary points: \( f'(x) = 12x(x + 3)(x - 2) = 0 \) when \( x = -3, x = 0, x = 2. \) Only \( x = 2 \) lies in \([1, \infty)\).

Finding coordinates:
\( x = 1: f(1) = -19; \) coordinates \((1, -19)\)
\( x = 2: f(2) = -54; \) coordinates \((2, -54)\)